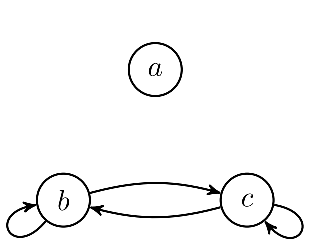
Suggested Solutions to Midterm Problems

1. (20 points)

   (a) Let $A = \{a, b, c, d, e, f\}$ and $R = \{(b, c), (d, e), (d, f)\}$, which is a binary relation on A.Give a symmetric and transitive but *not* reflexive binary relation on A that includes R. Please present the relation using a directed graph.

   Solution.



□

   (b) Suppose that R_1 and R_2 are equivalence relations on a set A. Is $R_1 \cup R_2$ an equivalence relation on A? Justify your answer.

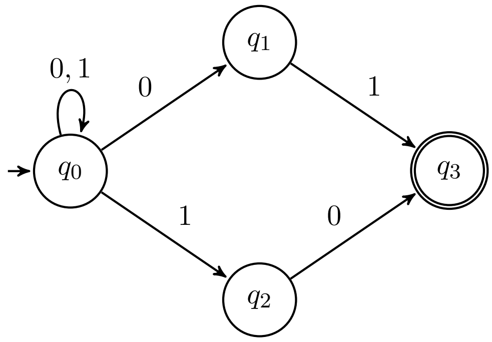
   Solution.
   Consider $R_1 = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$ and $R_1 = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}$, $R_1 \cup R_2$ would be $\{(a, a), (b, b), (c, c), (a, b), (b, a), (b, c), (c, b)\}$. Since (a, b) and (b, c) do not imply (a, c), $R_1 \cup R_2$ is not transitive. Therefore, when R_1 and R_2 are equivalence relations, $R_1 \cup R_2$ is not necessarily an equivalence relation.

□

2. (30 points)

   (a) Draw the state diagram of an NFA, with four states, that recognizes $\{w \in \{0, 1\}^* \mid w$ ends with 01 or 10$\}$.

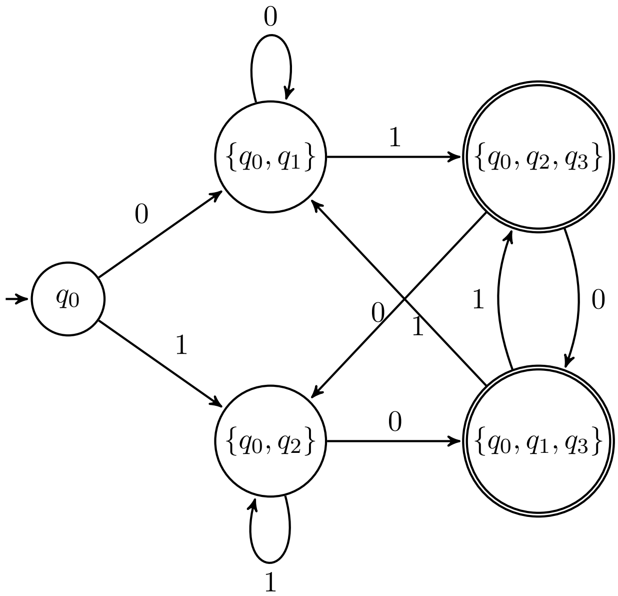
   Solution.



□

   (b) Convert the NFA in (a) into an equivalent DFA (using the procedure discussed in class); do not attempt to optimize the number of states, though you may omit the unreachable states.

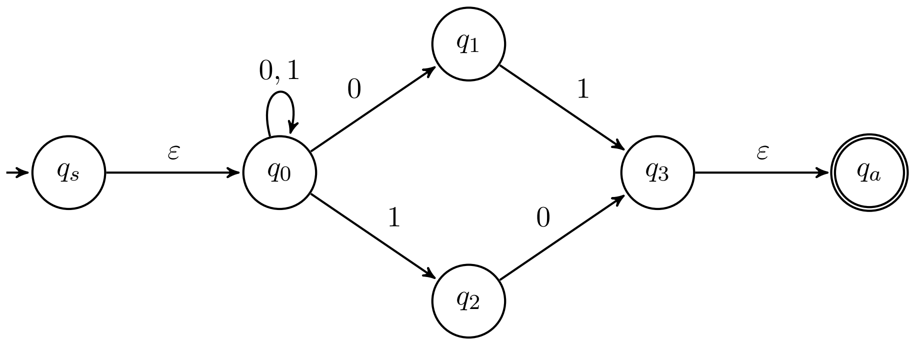
   Solution.



□

(c) Convert the NFA in (a) into an equivalent regular expression (using the procedure discussed in class).

Solution.



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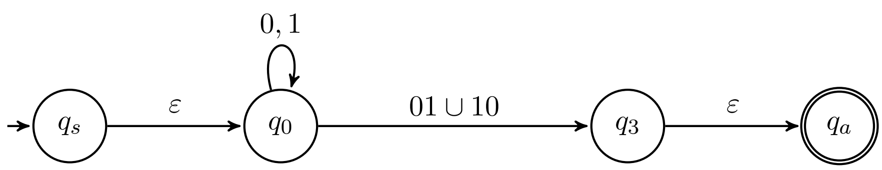

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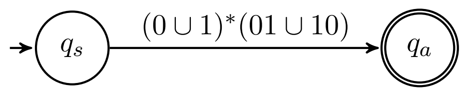


equivalent regular expression: $(0 \cup 1)^*(01 \cup 10)$ □

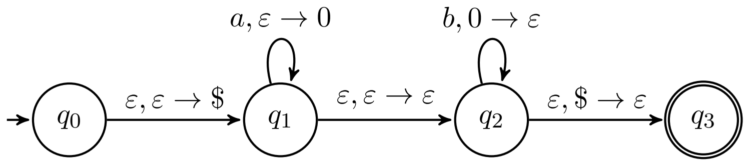
3. (20 points) Let $L = \{a^n b^n \mid n \in N\}$.

   (a) Prove or disprove L is regular.

   Solution.
   Suppose L is regular, we take s to be $a^p b^p$, where p is the pumping length. We divide s into xyz such that $|y| > 0$ and $|xy| \leq p$.
   Since $|xy| \leq p$, y falls in a^p. Consider $y = a^k$, $0 < k \leq p$. $xy^2z = a^{p+k}b^p$, which is $\notin L$. Since s does not belongs to L when we pump it. A contradiction occurs. Therefore, L is not regular. □

   (b) Prove or disprove L is context free.

   Solution.
   We construct a PDA that recognizes L to prove it is context-free:



   Since L can be recognized by some PDAs, it is context-free. □

4. (20 points) Consider the following CFG.

$$\begin{array}{lcl} E & \to & E + T \mid T \\ T & \to & T \times F \mid F \\ F & \to & (E) \mid a \end{array}$$

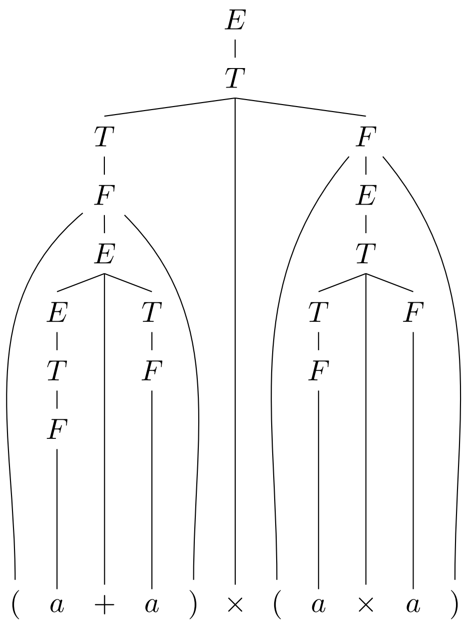
   (a) Give the (leftmost) derivation and parse tree for the string $(a + a) \times (a \times a)$.

   Solution.

The leftmost derivation:

$$\begin{array}{lcl} E & \Rightarrow & T \\ & \Rightarrow & T \times F \\ & \Rightarrow & F \times F \\ & \Rightarrow & (E) \times F \\ & \Rightarrow & (E + T) \times F \\ & \Rightarrow & (T + T) \times F \\ & \Rightarrow & (F + T) \times F \\ & \Rightarrow & (a + T) \times F \\ & \Rightarrow & (a + F) \times F \\ & \Rightarrow & (a + a) \times F \\ & \Rightarrow & (a + a) \times (E) \\ & \Rightarrow & (a + a) \times (T) \\ & \Rightarrow & (a + a) \times (T \times F) \\ & \Rightarrow & (a + a) \times (F \times F) \\ & \Rightarrow & (a + a) \times (a \times F) \\ & \Rightarrow & (a + a) \times (a \times a) \end{array}$$

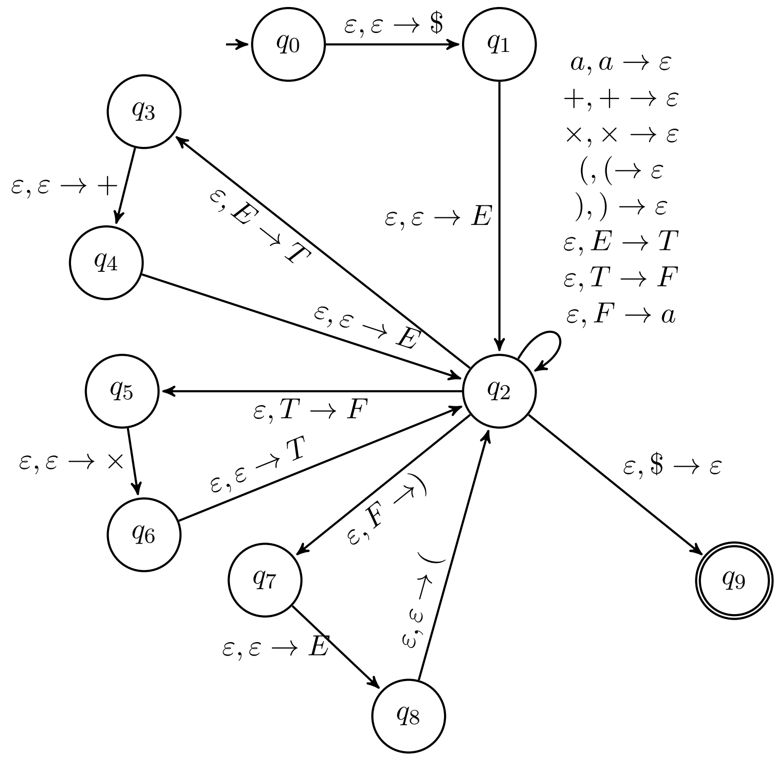
The parse tree:



□

(b) Convert the CFG in (a) into an equivalent PDA (using the procedure discussed in class).

Solution.



□

5. (10 points) Prove, using the pumping lemma, that $\{a^n b^n c^n \mid n \geq 1\}$ is not context free.

Solution.
Suppose $\{a^n b^n c^n \mid n \geq 1\}$ is context-free, we take s to be $a^p b^p c^p$, where p is the pumping length. There are three ways to divide s into $uvxyz$ such that $|vy| > 0$ and $|vxy| \leq p$:
Case1: vxy falls in a^p. When we pump up the string the number of a increases while the number of others remain. Since the numbers of each letter are unequal, the pumped string is not in $\{a^n b^n c^n \mid n \geq 1\}$.
Case2: vxy contains part of a^p and part of b^p. When we pump up the string the number of a and b increase while the number of c remains. Since the numbers of each letter are unequal, the pumped string is not in $\{a^n b^n c^n \mid n \geq 1\}$.
Case3: vxy falls in b^p. Similar to Caes1.
Case4: vxy contains part of b^p and part of c^p. Similar to Case2.
Case5: vxy falls in c^p. Similar to Caes1.
No matter how we divide s, the string no longer belongs to $\{a^n b^n c^n \mid n \geq 1\}$ when we pump it. A contradiction occurs. Therefore, $\{a^n b^n c^n \mid n \geq 1\}$ is not context free.

□