Homework Assignment #1

Note

This assignment is due 2:20PM Monday, March 4, 2019. Please write or type your answers on A4 (or similar size) paper. Put it on the instructor's desk on the due date before the class starts. Late submission will be penalized by 20% for each working day overdue. You may discuss the problems with others, but copying answers is strictly forbidden.

Problems

(Note: problems marked with "Exercise X.XX" or "Problem X.XX" are taken from [Sipser 2013] with probable adaptation.)

1. (Exercise 0.7; 30 points) For each part, give a binary relation that satisfies the condition. *Please illustrate the relation using a directed graph.*
   (a) Reflexive and transitive but not symmetric
   (b) Reflexive and symmetric but not transitive
   (c) Symmetric and transitive but not reflexive
2. (20 points) For each part, determine whether the binary relation on the set of integers is an equivalence relation. If it is, please provide a proof; otherwise, please give a counterexample.
   (a) The two numbers have a common divisor.
   (b) For a fixed non-zero divisor, the two numbers have the same remainder. (Note: suppose 2 is the divisor. 4 and 6 have the same remainder, while 4 and 5 do not.)
3. (Problem 0.10; 20 points) Show that every graph without self-loop edge and having two or more nodes contains two nodes with the same degree.
4. (Problem 0.11; 20 points) Find the error in the following proof that all horses are the same color.

   CLAIM: In any set of h horses, all horses are the same color.

   PROOF: By induction on h.

   Basis ($h = 1$): In any set containing just one horse, all horses clearly are the same color.

   Induction step ($h > 1$): We assume that the claim is true for $h = k$ ($k \geq 1$) and prove that it is true for $h = k + 1$. Take any set H of $k + 1$ horses. We show that all the horses in this set are the same color. Remove one horse from this set to obtain the set H_1 with just k horses. By the induction hypothesis, all the horses in H_1 are the same color. Now replace the removed horse and remove a different one to obtain the set H_2. By the same argument, all the horses in H_2 are the same color. Therefore all the horses in H must be the same color, and the proof is complete.

5. (Problem 0.13; 10 points) Find the error in the following proof that $2 = 1$.

   Consider the equation $a = b$. Multiply both sides by a to obtain $a^2 = ab$. Subtract b^2 from both sides to get $a^2 - b^2 = ab - b^2$. Now factor each side, $(a+b)(a-b) = b(a-b)$, and divide each side by $(a-b)$ to get $a+b = b$. Finally, let a and b equal 1, which shows that $2 = 1$.