Theory of Computing
Turing Machines

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(original created by Bow-Yaw Wang, some slides from Yih-Kuen Tsay)

Schematic of Turing Machines

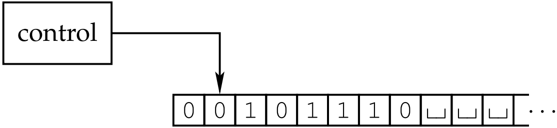


Figure: Schematic of Turing Machines

- A Turing machine has a finite set of control states.
- A Turing machine reads and writes symbols on an infinite tape.
- A Turing machine starts with an input on the left end of the tape.
- A Turing machine moves its read-write head in both directions.
- A Turing machine outputs accept or reject by entering its accepting or rejecting states respectively.
    - A Turing machine need not read all input symbols.
    - A Turing machine may not accept nor reject an input.

- Consider $B = \{w\#w : w \in \{\mathtt{0}, \mathtt{1}\}^*\}$.
- M_1 = "On input string w:
  1. Record the first uncrossed symbol from the left and cross it. If the first uncrossed symbol is #, go to step 6.
  2. Move the read-write head to the symbol #. If there is no such symbol, reject.
  3. Move to the first uncrossed symbol to the right.
  4. Compare with the symbol recorded at step 1. If they are not equal, reject.
  5. Cross the current symbol and go to step 1.
  6. Check if all symbols to the right of # are crossed. If so, accept; otherwise, reject."

Turing Machines – Formal Definition

Definition 1

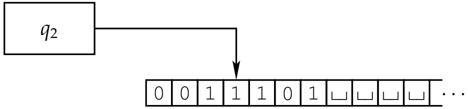
A Turing machine is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$ where

- Q is the finite set of states;
- Σ is the finite input alphabet not containing the blank symbol $\sqcup$;
- Γ is the finite tape alphabet with $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$;
- $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$ is the transition function;
- $q_0 \in Q$ is the start state;
- $q_{\text{accept}} \in Q$ is the accept state; and
- $q_{\text{reject}} \in Q$ is the reject state with $q_{\text{reject}} \neq q_{\text{accept}}$.

- We only consider deterministic Turing machines.
- Initially, a Turing machine receives its input $w = w_1 w_2 \cdots w_n \in \Sigma^*$ on the leftmost n cells of the tape.
- Other cells on the tape contain the blank symbol $\sqcup$.

Computation of Turing Machines

- A configuration of a Turing machine contains its current states, current tape contents, and current head location.
- Let $q \in Q$ and $u, v \in \Gamma$. We write uqv to denote the configuration where the current state is q, the current tape contents is uv, and the current head location is the first symbol of v.
  - When we say "the current tape contents is uv," we mean an infinite tape contains $uv \sqcup \sqcup \cdots \sqcup \cdots$.
- Consider the configuration $\texttt{001}q_2\texttt{1101}$. The Turing machine
  - is at the state q_2;
  - has the tape contents `0011101`; and
  - has its head location at the second `1` from the left.



Computation of Turing Machines

- Let C_1 and C_2 be configurations. We say C_1 yields C_2 if the Turing machine can go from C_1 to C_2 in one step.
- Formally, let $a, b, c \in \Gamma$, $u, v \in \Gamma^*$, and $q_i, q_j \in Q$.

$$\begin{array}{rl} uaq_ibv \ \underline{\text{yields}}\ uq_jacv & \text{if } \gamma(q_i, b) = (q_j, c, L) \\ q_ibv \ \underline{\text{yields}}\ q_jcv & \text{if } \gamma(q_i, b) = (q_j, c, L) \\ uaq_ibv \ \underline{\text{yields}}\ uacq_jv & \text{if } \gamma(q_i, b) = (q_j, c, R) \end{array}$$

- Note the special case when the current head location is the leftmost cell of the tape.
  - A Turing machine updates the leftmost cell without moving its head.
- Recall that uaq_i is in fact $uaq_i\sqcup$.

Accept, Reject, and Halting

- The start configuration of M on input w is q_0w.
- An accepting configuration of M is a configuration whose state is q_{accept}.
- A rejecting configuration of M is a configuration whose state is q_{reject}.
- Accepting and rejecting configurations are halting configurations and do not yield further configurations.
  - That is, a Turing machine accepts or rejects as soon as it reaches an accepting or rejecting configuration.

Recognizable Languages

- A Turing machine M accepts an input w if there is a sequence of configurations $C_1, C_2, \ldots, C_k$ such that
  - C_1 is the start configuration of M on input w;
  - each C_i yields C_{i+1}; and
  - C_k is an accepting configuration.
- The language of M or the language recognized by M (written $L(M)$) is thus

$$L(M) = \{w : M \text{ accepts } w\}.$$

Definition 2

A language is Turing-recognizable or recursively enumerable if some Turing machine recognizes it.

Decidable Languages

- When a Turing machine is processing an input, there are three outcomes: accept, reject, or loop.
  - "Loop" means it never enters a halting configuration.
- A deterministic finite automaton or deterministic pushdown automaton have only two outcomes: accept or reject.
- For a nondeterministic finite automaton or nondeterminsitic pushdown automaton, it can also loop.
  - "Loop" means it does not finish reading the input (ϵ-transitions).
- A Turing machine that halts on all inputs is called a decider.
- When a decider recognizes a language, we say it decides the language.

Definition 3

A language is Turing-decidable (decidable, or recursive) if some Turing machine decides it.

Turing Machines – Example M_2

- $A = \{0^{2^n} \mid n \geq 0\}$.
- A decider M_2 for A can be defined to work as follows:
  1. Sweep left to right across the tape, crossing off every second 0.
  2. If in stage 1 the tape contained a single 0, accept.
  3. If in stage 1 the tape contained more than one 0 and the number of 0s was odd, reject.
  4. Return head to the left-hand end of the tape.
  5. Go to stage 1.

Turing Machines – Example M_2 (cont.)

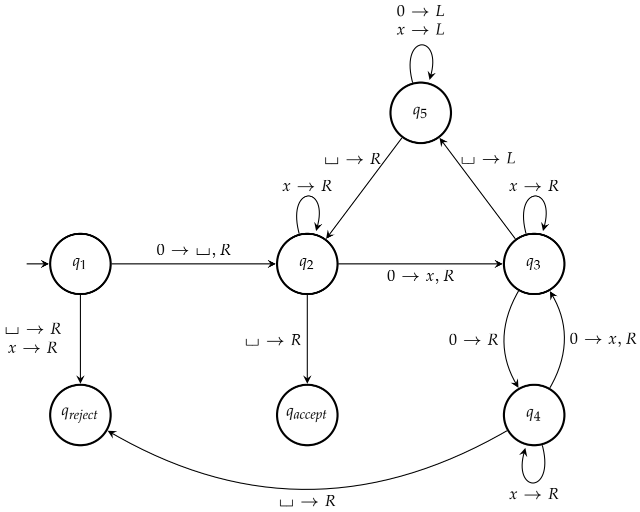
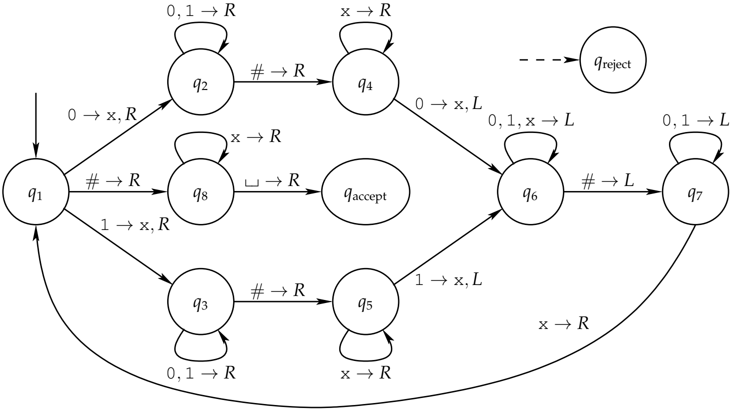


Figure: Turing Machine M_2

Turing Machines – Example M_1

- We now formally define $M_1 = (Q, \Sigma, \Gamma, \delta, q_1, q_{\text{accept}}, q_{\text{reject}})$ which decides $B = \{w\#w : w \in \{\mathtt{0}, \mathtt{1}\}^*\}$.
- $Q = \{q_1, \ldots, q_8, q_{\text{accept}}, q_{\text{reject}}\}$;
- $\Sigma = \{\mathtt{0}, \mathtt{1}, \#\}$ and $\Gamma = \{\mathtt{0}, \mathtt{1}, \#, \mathtt{x}, \sqcup\}$.



Exercise

- In each of the parts, give the sequence of configurations that M_2 enters when started on the indicated input string.
  - 0.
  - 00.
- In each of the parts, give the sequence of configurations that M_1 enters when started on the indicated input string.
  - 11.
  - 1#1.

Turing Machines – Example M_3

- $C = \{a^i b^j c^k \mid i \times j = k \text{ and } i, j, k \geq 1\}$.
- A decider M_3 for C:
  1. Scan the input to be sure that it is a member of $aa^*bb^*cc^*$ and reject if it isn't.
  2. Return the head to the left-hand end of the tape.
  3. Cross off an a and scan to the right until a b occurs. Shuttle between the b's and c's, crossing off one of each until all b's are gone.
  4. Restore the crossed off b's and repeat Stage 3 if there is another a to cross off.
  5. If all a's and c's are crossed off, accept; otherwise, reject.

Turing Machines – Example M_4

- $E = \{\#x_1\#x_2\#\cdots\#x_l \mid x_i \in \{0,1\}^* \text{ and } x_i \neq x_j \text{ (for} i \neq j)\}$.
- A decider M_4 for E:
  1. Place a mark on top of the leftmost tape symbol. If that symbol was not a #, reject.
  2. Scan right to the next # and place a second mark on top of it. If no # occurs before a blank, accept.
  3. Compare, by zig-zagging, the two strings to the right of the marked #'s. If they are equal, reject.
  4. Move the second mark to the next # symbol. If not doable, move the first mark to the next # to its right and the second mark to the # after that. If not doable, accept.
  5. Go to Stage 3.

Turing Machines whose Heads can Stay

- Recall that the transition function of a Turing machine indicate whether its read-write head moves left or right.
- Consider a new Turing machine whose head can stay.
- Hence we have $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R, S\}$.
- Is the new Turing machine more powerful?
- Of course not, we can always simulate S by an R and then an L.

Multitape Turing Machines

- A multitape Turing machine has several tapes.
- Initially, the input appears on the tape 1.
- If a multitape Turing machine has k tapes, its transition function now becomes

$$\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R\}^k$$

- $\delta(q_i, a_1, \ldots, a_k) = (q_j, b_1, \ldots, b_k, d_1, \ldots, d_k)$ means that if the machine is in state q_i and reads a_i from tape i for $1 \leq i \leq k$, it goes to state q_j, writes b_i to tape i for $1 \leq i \leq k$, and moves the tape head i towards the direction d_i for $1 \leq i \leq k$.
- Are multitape Turing machines more powerful than signel-tape Turing machines?

Multitape Turing Machines

Theorem 4

Every multitape Turing machine has an equivalent single-tape Turing machine.

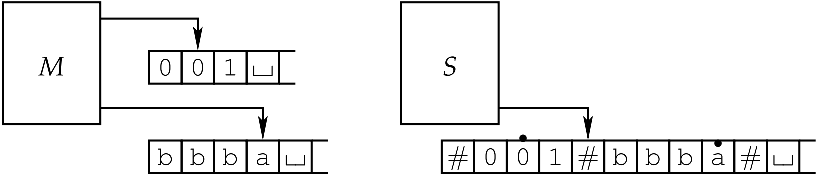
Proof.

We use a special new symbol # to separate contents of k tapes. Moreover, k marks are used to record locations of the k virtual heads.

$S =$ "On input $w = w_1 w_2 \cdots w_n$:

1. Write w in the correct format: $\#\dot{w}_1 w_2 \cdots w_n \# \dot{\sqcup} \# \dot{\sqcup} \# \cdots \#$.
2. Scan the tape and record all symbols under virtual heads. Then update the symbols and virtual heads by the transition function of the k-tape Turing machine.
3. If S moves a virtual head to the right onto a #, S writes a blank symbol and shifts the tape contents from this cell to the rightmost # one cell to the right. Then S resumes simulation." □

Multitape Turing Machines



- A "mark" is in fact a different tape symbol.
  - Say the tape alphabet of the multitape TM M is $\{\mathtt{0}, \mathtt{1}, \mathtt{a}, \mathtt{b}, \sqcup\}$.
  - Then S has the tape alphabet $\{\#, \mathtt{0}, \mathtt{1}, \mathtt{a}, \mathtt{b}, \sqcup, \dot{\mathtt{0}}, \dot{\mathtt{1}}, \dot{\mathtt{a}}, \dot{\mathtt{b}}, \dot{\sqcup}\}$.

Corollary 5

A language is Turing-Recognizable if and only if some multitape Turing machine recognizes it.

Nondeterministic Turing Machines

- A nondeterministic Turing machine has its transition function of type $\delta : Q \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{L, R\})$.
- Is nondeterministic Turing machines more powerful than deterministic Turing machines?
  - Recall that nondeterminism does not increase the expressive power in finite automata.
  - Yet nondeterminism does increase the expressive power in pushdown automata.

Nondeterministic Turing Machines

Theorem 6

Every nondeterministic Turing machine has an equivalent deterministic Turing machine.

Proof.

Nondeterministic computation can be seen as a tree. The root is the start configuration. The children of a tree node are all possible configurations yielded by the node. By ordering children of a node, we associate an address with each node. For instance, ϵ is the root; $\mathtt{1}$ is the first child of the root; $\mathtt{21}$ is the first child of the second child of the root.
We simulate an NTM N with a 3-tape DTM D. Tape 1 contains the input; tape 2 is the working space; and tape 3 records the address of the current configuration.
Let b be the maximal number of choices allowed in N. Define $\Sigma_b = \{\mathtt{1}, \mathtt{2}, \ldots, \mathtt{b}\}$. We now describe the Turing machine D.

Nondeterministic Turing Machines

Proof.

1. Initially, tape 1 contains the input w; tape 2 and 3 are empty.
2. Copy tape 1 to tape 2.
3. Simulate N from the start state on tape 2 according to the address on tape 3.
   - When compute the next configuration, choose the transition by the next symbol on tape 3.
   - If no more symbol is on tape 3, the choice is invalid, or a rejecting configuration is yielded, go to step 4.
   - If an accepting configuration is yielded, accept the input.
4. Replace the string on tape 3 with the next string lexicographically and go to step 2. □

- Observe the D simulates N by breadth.
  - Can we simulate by depth?

Nondeterministic Turing Machines

Corollary 7

A language is Turing-recognizable if and only if some nondeterministic Turing machine recognizes it.

- A nondeterministic Turing machine is a decider if all branches halt on all inputs.
- If the NTM N is a decider, a slight modification of the proof makes D always halt. (How?)

Corollary 8

A language is decidable if and only if some nondeterministic Turing machine decides it.

Schematic of Enumerators

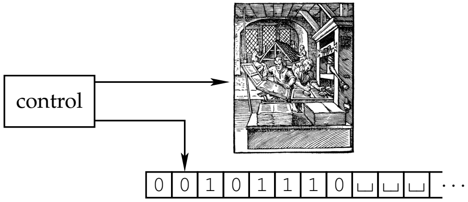


Figure: Schematic of Enumerators

- An enumerator is a Turing machine with a printer.
- An enumerator starts with a blank input tape.
- An enumerator outputs a string by sending it to the printer.
- The language enumerated by an enumerator is the set of strings printed by the enumerator.
  - Since an enumerator may not halt, it may output an infinite number of strings.
  - An enumerator may output the same string several times.

Enumerators

Theorem 9

A language is Turing-recognizable if and only if some enumerator enumerates it.

Proof.

Let E be an enumerator. Consider the following TM M:

$M =$ "On input w :

1. Run E and compare any output string with w.
2. Accept if E ever outputs w."

Conversely, let M be a TM recognizing A. Consider

$E =$ "Ignore the input.

1. Repeat for $i = 1, 2, \ldots$
   1. Let $s_1, s_2, \ldots, s_i$ be the first i strings in Σ^* (say, lexicographically).
   2. Run M for i steps on each of $s_1, s_2, \ldots, s_i$.
   3. If M accepts s_j for $1 \leq j \leq i$, output s_j.

$\square$

Exercise

- Give a formal definition of an enumerator. Consider it to be a type of two-tape Turing machine that uses its second tape as the printer.
- Give implementation-level description of Turing machines that decide the following language over the alphabet $\{0, 1\}$.

$$\{w \mid w \text{ contains an equal number of } 0s \text{ and } 1s\}$$

Algorithms

- Let us suppose we lived before the invention of computers.
  - say, circa 300 BC, around the time of Euclid.
- Consider the following problem:
  Given two positive integers a and b, find the largest integer r such that r divides a and r divides b.
- How do we "find" such an integer?
- Euclid's method is in fact an algorithm.
- Keep in mind that the concept of algorithms has been in mathematics long before the advent of computer science.

Hilbert's Problems

- Mathematician David Hilbert listed 23 problems in 1900.
  - These problems are challenges for mathematicians in 20th century.
- His 10th problem is to devise "a process according to which it can be determined by a finite number of operations," that tests whether a polynomial has an integral root.
  - In other words, Hilbert wants to find an algorithm to test whether a polynomial has an integral root.
- If such an algorithm exists, we just need to invent it.
- What if there is no such algorithm?
  - How can we argue Hilbert's 10th problem has no solution?
- We need a precise definition of algorithms!

Church-Turing Thesis

- In 1936, two papers came up with definitions of algorithms.
- Alonzo Church used λ-calculus to define algorithms.
  - If you don't know λ-calculus, take Programming Languages.
- Alan Turing used Turing machines to define algorithms.
  - If you don't know TM now, please consider dropping this course.
- It turns out that both definitions are equivalent!
- The connection between the informal concept of algorithms and the formal definitions is called the Church-Turing thesis.

Hilbert's 10th Problem

- In 1970, Yuri Matijasevič showed that Hilbert's 10th problem is not solvable.
  - That is, there is no algorithm for testing whether a polynomial has an integral root.
- Define $D = \{p : p \text{ is a polynomial with an integral root}\}$.
- Consider the following TM:
  $M =$ "The input is a polynomial p over variables $x_1, x_2, \ldots, x_k$
  1. Evaluate p on an enumeration of k-tuple of integers.
  2. If p ever evaluates to 0, accept."
- M recognizes D but does not decide D.

Format for Turing Machines

- For any object (numbers, polynomials, graphs, etc) O, $\langle O \rangle$ represents an encoding of O as a string.
  - If we have several objects $O_1, O_2, \ldots, O_k$, their encoding into a single string is denoted by $\langle O_1, O_2, \ldots, O_k \rangle$.
- To describe a Turing machine M, we use the following format: $M =$ "On input $\langle O \rangle$, the encoding of O :
  1. stage 1.
  2. stage 2.

  etc."
- The TM M implicitly checks if $\langle O \rangle$ is a proper encoding of O. If not, M rejects immediately.

Format for Turing Machines

Example 10

Let $A = \{\langle G \rangle : G$ is a connected undirected graph $\}$. Describe a TM deciding A.

Proof.

$M =$ "On input $\langle G \rangle$, the encoding of a graph G:

1. Select the first node of G and mark it.
2. Repeat until no new node is marked:
   1. For each node in G, mark it if there is an edge connecting the node to a marked node.
3. Check if all nodes of G are marked. If yes, *accept*; otherwise, *reject*." □

- Double quotes (" and ") mean the description is informal.
  - Yet we are confident that it corresponds to a formal description.